ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.841174

INTERNATIONAL JOURNAL OF RESEARCH – GRANTHAALAYAH
A knowledge Repository

Science

ON THE POSITIVE PELL EQUATION $Y^2 = 72X^2 + 36$

M.A.Gopalan [1], A.Kavitha [2], A.Jesintha Mary [3]
[1] Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, India
[2] Assistant Professor, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, India
[3] M.Phil Scholar, Department of Mathematics, Shrimati Indira Gandhi College, Trichy-620 002, India

Abstract

The binary quadratic equation represented by the positive pellian $y^2 = 72x^2 + 36$ is analysed for its distinct integer solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.

Keywords: Binary Quadratic; Hyperbola; Parabola; Integral Solutions; Pell Equation.

2010 Mathematics Subject Classification: 11D09.

Cite This Article: M.A.Gopalan, A.Kavitha, and A.Jesintha Mary. (2017). "ON THE POSITIVE PELL EQUATION $Y^2 = 72X^2 + 36$." *International Journal of Research - Granthaalayah*, 5(7:SE), 68-74. https://doi.org/10.5281/zenodo.841174.

1. Introduction

The binary quadratic equation of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been studied by various mathematicians for its non-trivial integral solutions when D takes different values [1-4]. For an extensive review of various problems, one may refer [5-20]. In this communication, yet another interesting hyperbola given by $y^2 = 72x^2 + 36$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

2. Method of Analysis

Consider the binary quadratic equation

$$y^2 = 72x^2 + 36 \tag{1}$$

whose smallest positive integer solution is $x_0 = 2, y_0 = 18$

ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.841174

To obtain the other solutions of (1), consider the pell equation $y^2 = 72x^2 + 1$ whose solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{72}} g_n, \tilde{y}_n = \frac{1}{2} f_n \qquad (2)$$

where,

$$f_n = \left(17 + 2\sqrt{72}\right)^{n+1} + \left(17 - 2\sqrt{72}\right)^{n+1}$$

$$g_n = \left(17 + 2\sqrt{72}\right)^{n+1} - \left(17 - 2\sqrt{72}\right)^{n+1}, n = 0,1,2,3,....$$

Applying Brahamagupta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$\sqrt{72} x_{n+1} = \sqrt{72} f_n + 9 g_n$$

$$y_{n+1} = 9 f_n + \sqrt{72} g_n$$

The recurrence relations satisfied by the solutions (2) are given by

$$x_{n+3} - 34 x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 34 y_{n+2} + y_{n+1} = 0$$

Some numerical examples of x & y satisfying (1) are given in the Table 1 below:

Table 1: Examples

n	x_n	y_n
0	2	18
1	70	594
2	2378	20178
3	80782	685458
4	2744210	23285394

From the above table, we observe some interesting relations among the solutions which are presented below:

1) x_n and y_n values are always even.
2) Each of the following expressions is a nasty number:

- $3x_{2n+3} - 99x_{2n+2} + 12$
- $\frac{3x_{2n+4} - 3363x_{2n+2} + 408}{34}$
- $\frac{6y_{2n+3} - 1680x_{2n+2} + 204}{17}$
- $\frac{6y_{2n+4} - 57072x_{2n+2} + 6924}{577}$
- $\frac{198y_{2n+2} - 48x_{2n+3} + 204}{17}$
- $198y_{2n+3} - 1680x_{2n+3} + 12$

 ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.841174

- $\frac{198y_{2n+4} - 57072x_{2n+3} + 204}{17}$
- $\frac{6726y_{2n+2} - 48x_{2n+4} + 6924}{577}$
- $\frac{6726y_{2n+3} - 1680x_{2n+4} + 204}{17}$
- $6726y_{2n+4} - 57072x_{2n+4} + 12$
- $99x_{2n+4} - 3363x_{2n+3} + 12$
- $\frac{35y_{2n+2} - y_{2n+3} + 36}{3}$
- $\frac{1189y_{2n+2} - y_{2n+4} + 1224}{102}$
- $\frac{1189y_{2n+3} - 35y_{2n+4} + 36}{3}$

3) Each of the following expressions is a cubical integer:

- $4[x_{3n+4} - 33x_{3n+3} + 12(x_{n+2} - 33x_{n+1})]$
- $4624[x_{3n+5} - 1121x_{3n+3} + 3(x_{n+3} - 1121x_{n+1})]$
- $23409[9y_{3n+4} - 2520x_{3n+3} + 3(9y_{n+2} - 2520x_{n+1})]$
- $26967249[9y_{3n+5} - 85608x_{3n+3} + 3(9y_{n+3} - 85608x_{n+1})]$
- $23409[297y_{3n+3} - 72x_{3n+4} + 3(297y_{n+1} - 72x_{n+2})]$
- $3[297y_{3n+4} - 2520x_{3n+4} + 3(297y_{n+2} - 2520x_{n+2})]$
- $23409[297y_{3n+5} - 85608x_{3n+4} + 3(297y_{n+3} - 85608x_{n+2})]$
- $26967249[10089y_{3n+3} - 72x_{3n+5} + 3(10089y_{n+1} - 72x_{n+3})]$
- $23409[10089y_{3n+4} - 2520x_{3n+5} + 3(10089y_{n+2} - 2520x_{n+3})]$
- $3[10089y_{3n+5} - 85608x_{3n+5} + 3(10089y_{n+3} - 85608x_{n+3})]$
- $324[297x_{3n+5} - 10089x_{3n+4} + 3(297x_{n+3} - 10089x_{n+2})]$
- $324[35y_{3n+3} - y_{3n+4} + 3(35y_{n+1} - y_{n+2})]$
- $374544[1189y_{3n+3} - y_{3n+5} + 3(1189y_{n+1} - y_{n+3})]$
- $324[1189y_{3n+4} - 35y_{3n+5} + 3(1189y_{n+2} - 35y_{n+3})]$

4) Relations among the solutions:

- $18x_{n+3} = 612x_{n+2} - 18x_{n+1}$
- $18y_{n+1} = 9x_{n+2} - 153x_{n+1}$
- $18y_{n+2} = 153x_{n+2} - 9x_{n+1}$
- $18y_{n+3} = 5193x_{n+2} - 153x_{n+1}$
- $612y_{n+1} = 9x_{n+3} - 5193x_{n+1}$

 ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.841174

- $612y_{n+2} = 153x_{n+3} - 153x_{n+1}$
- $612y_{n+3} = 5193x_{n+3} - 9x_{n+1}$
- $153y_{n+1} = 9y_{n+2} - 1296x_{n+1}$
- $5193y_{n+1} = 9y_{n+3} - 44064x_{n+1}$
- $5193y_{n+2} = 153y_{n+3} - 1296x_{n+1}$
- $9y_{n+1} = 153y_{n+2} - 1296x_{n+2}$
- $153y_{n+1} = 153y_{n+3} - 44064x_{n+2}$
- $153y_{n+2} = 9y_{n+3} - 1296x_{n+2}$
- $153y_{n+3} = 9y_{n+2} + 1296x_{n+3}$
- $9x_{n+2} = -18y_{n+3} + 153x_{n+3}$
- $9y_{n+1} = 5193y_{n+3} - 44064x_{n+3}$
- $18y_{n+1} = 153x_{n+3} - 5193x_{n+2}$
- $18y_{n+2} = 9x_{n+3} - 153x_{n+2}$
- $1296x_{n+3} = -153y_{n+1} + 5193y_{n+2}$
- $18y_{n+3} = -18y_{n+1} + 612y_{n+2}$

3. Remarkable Observation

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the Table 2 below:

Table 2: Hyperbolas

S.no	(X,Y)	Hyperbola
1	$(35x_{n+1} - x_{n+2}, x_{n+2} - 33x_{n+1})$	$81Y^2 - 72X^2 = 1296$
2	$(1189x_{n+1} - x_{n+3}, x_{n+3} - 1121x_{n+1})$	$81Y^2 - 72X^2 = 1498176$
3	$(297x_{n+1} - y_{n+2}, 9y_{n+2} - 2520x_{n+1})$	$Y^2 - 72X^2 = 93636$
4	$(10089x_{n+1} - y_{n+3}, 9y_{n+3} - 85608x_{n+1})$	$Y^2 - 72X^2 = 107868996$
5	$(9x_{n+2} - 35y_{n+1}, 297y_{n+1} - 72x_{n+2})$	$Y^2 - 72X^2 = 93636$
6	$(297x_{n+2} - 35y_{n+2}, 297y_{n+2} - 2520x_{n+2})$	$Y^2 - 72X^2 = 324$
7	$(10089x_{n+2} - 35y_{n+3}, 297y_{n+3} - 85608x_{n+2})$	$Y^2 - 72X^2 = 93636$
8	$(9x_{n+3} - 1189y_{n+1}, 10089y_{n+1} - 72x_{n+3})$	$Y^2 - 72X^2 = 107868996$
9	$(297x_{n+3} - 1189y_{n+2}, 10089y_{n+2} - 2520x_{n+3})$	$Y^2 - 72X^2 = 93636$
10	$(10089x_{n+3} - 1189y_{n+3}, 10089y_{n+3} - 85608x_{n+3})$	$Y^2 - 72X^2 = 324$
11	$(1189x_{n+2} - 35x_{n+3}, 297x_{n+3} - 10089x_{n+2})$	$Y^2 - 72X^2 = 1296$

 ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.841174

12	$\left(y_{n+2}-33y_{n+1}, 35y_{n+1}-y_{n+2}\right)$	$288Y^2-324X^2=373248$
13	$\left(y_{n+3}-1121y_{n+1}, 1189y_{n+1}-y_{n+3}\right)$	$2312Y^2-2601X^2=$ 3463782912
14	$\left(297y_{n+3}-10089y_{n+2}, 1189y_{n+2}-35y_{n+3}\right)$	$72Y^2-X^2=93312$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the Table 3 below:

Table 3: Parabolas

S.no	(X, Y)	**Parabola**
1	$\left(35x_{n+1}-x_{n+2}, x_{2n+3}-33x_{2n+2}\right)$	$72X^2=162Y-648$
2	$\left(1189x_{n+1}-x_{n+3}, x_{2n+4}-1121x_{2n+2}\right)$	$72X^2=5508Y-749088$
3	$\left(297x_{n+1}-y_{n+2}, 9y_{2n+3}-2520x_{2n+2}\right)$	$72X^2=153Y-46818$
4	$\left(10089x_{n+1}-y_{n+3}, 9y_{2n+4}-85608x_{2n+2}\right)$	$72X^2=5193Y-53934498$
5	$\left(9x_{n+2}-35y_{n+1}, 297y_{2n+2}-72x_{2n+3}\right)$	$72X^2=153Y-46818$
6	$\left(297x_{n+2}-35y_{n+2}, 297y_{2n+3}-2520x_{2n+3}\right)$	$72X^2=9Y-162$
7	$\left(10089x_{n+2}-35y_{n+3}, 297y_{2n+4}-85608x_{2n+3}\right)$	$72X^2=153Y-46818$
8	$\left(9x_{n+3}-1189y_{n+1}, 10089y_{2n+2}-72x_{2n+4}\right)$	$72X^2=5193Y-539344$
9	$\left(297x_{n+3}-1189y_{n+2}, 10089y_{2n+3}-2520x_{2n+4}\right)$	$72X^2=153Y-46818$
10	$\left(\begin{matrix}10089x_{n+3}-1189y_{n+3}, \\ 10089y_{2n+4}-85608x_{2n+4}\end{matrix}\right)$	$72X^2=9Y-162$
11	$\left(1189x_{n+2}-35x_{n+3}, 297x_{2n+4}-10089x_{2n+3}\right)$	$72X^2=18Y-648$
12	$\left(y_{n+2}-33y_{n+1}, 35y_{2n+2}-y_{2n+3}\right)$	$X^2=16Y-576$
13	$\left(y_{n+3}-1121y_{n+1}, 1189y_{2n+2}-y_{2n+4}\right)$	$X^2=544Y-665856$
14	$\left(297y_{n+3}-10089y_{n+2}, 1189y_{2n+3}-35y_{2n+4}\right)$	$X^2=1296Y-46656$

III. Consider $m=x_{n+1}+y_{n+1}, n=x_{n+1}$. Observe that $m>n>0$. Treat m, n as the generators of the Pythagorean Triangle $T(\alpha,\beta,\gamma)$, where $\alpha=2mn$, $\beta=m^2-n^2$, $\gamma=m^2+n^2$

Then the following interesting relations are observed:

a) $\alpha-36\beta+35\gamma=-36$

b) $37\alpha-\gamma+\dfrac{144A}{P}=36$

c) $-34\gamma+36\beta-38\alpha+\dfrac{144A}{P}=72$

 ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.841174

d) $\frac{2A}{P} = x_{n+1}y_{n+1}$

4. Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the positive Pell equation $y^2 = 72x^2 + 36$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of positive Pell equations and determine their integer solutions along with suitable properties.

References

[1] Dickson L.E., "History of theory of Number", Chelsa Publishing Company, (vol-2), Newyork, 1952.
[2] Mordel L.J ., "Diophantine Equations", Academic Press, New york, 1969.
[3] Telang S.J., "Number theory", Tata McGraw Hill Publishing Company Limited, New Delhi, 2000.
[4] David M.Burton., "Elementary Number Theory", Tata McGraw Hill Publishing Company Limited, New Delhi, 2002.
[5] Gopalan M.A., Vidhyalakshmi S and Devibala S, "On the Diophantine Equation $3x^2 + xy = 14$" Acta Cinecia Indica, 2007, vol-XXXIIIM, No.2, 645-648.
[6] Gopalan M.A and Janaki G, "Observations on $y^2 = 3x^2 + 1$", Acta Cinecia Indica, vol-XXIVM, 2008, No.2, 693-696.
[7] Gopalan M.A and Sangeetha G, "A Remarkable Observation on $y^2 = 10x^2 + 1$", Impact Journal of science and Technology, 2010, 4(1), 103-106.
[8] Gopalan M.A and Vijayalakshmi R, "Observations on the integral solutions of $y^2 = 5x^2 + 1$", Impact Journal of science and Technology, 2010, 4(4), 125-129.
[9] Gopalan M.A and Sivakami B, "Observations on the integral solutions of $y^2 = 7x^2 + 1$", Antarctica Journal of Mathematics, 2010, 7(3), 291-296.
[10] Gopalan M.A and Yamuna R.S, "Remarkable Observations on the binary quadratic equation $y^2 = (k^2 + 1)x^2 + 1, k \in z - \{0\}$", Impact Journal of science and Technology, 2010, 4(4), 61-65.
[11] Gopalan M.A and Vijayalakshmi R, "Special Pythagorean triangle generated through the integral solutions of the equation $y^2 = (k^2 - 1)x^2 + 1$", Antarctica Journal of Mathematics, 2010, 7(5), 503-507.
[12] Gopalan M.A and Srividhya G, "Relations among M-gonal Number through the equation $y^2 = 2x^2 + 1$", Antarctica Journal of Mathematics, 2010, 7(3), 363-369.
[13] Gopalan M.A and Palanikumar R, "Observation on $y^2 = 12x^2 + 1$", Antarctica Journal of Mathematics, 2011, 8(2), 149-152.
[14] Gopalan M.A., Vidhyalakshmi S., Usha Rani T.R and Mallika S, "Observations on $y^2 = 12x^2 - 3$", Bessel Journal of Math, 2012 ,2(3), 153-158.
[15] Gopalan M.A., Vidhyalakshmi S., Umarani J, "Remarkable Observations on the hyperbola $y^2 = 24x^2 + 1$", Bulletin of Mathematics and statistics Research, 2013, 1, 9-12.

ISSN- 2350-0530(O), ISSN- 2394-3629(P)
DOI: 10.5281/zenodo.841174

[16] Gopalan M.A., Vidhyalakshmi S., Maheswari D, "Observations on the hyperbola $y^2 = 30x^2 + 1$", International Journal of Engineering Research, 2013, 1(3), 312-314.
[17] Geetha T., Gopalan M.A., Vidhyalakshmi S, "Observations on the hyperbola $y^2 = 72x^2 + 1$", Scholars Journal of physics, Mathematics and statistics, 2014, 1(1), 1-3 .
[18] Gopalan M.A and Sivakami B, "Special Pythagorean triangle generated through the integral solutions of the equation $y^2 = (k^2 + 2k)x^2 + 1$", Diaphantus. J.math., 2(1), 2013, 25-30.
[19] Gopalan M.A., Vidhyalakshmi S., Usha Rani T.R., Agalya K, "Observations on the hyperbola $y^2 = 110x^2 + 1$", International Journal of multidisciplinary Research and Development, 2015, vol 2, issue 3, 237-239.
[20] Vidhyalakshmi S., Gopalan M.A., Sumithra S., Thiruniraiselvi N, "Observations on the hyperbola $y^2 = 60x^2 + 4$", JIRT, 2014, vol 1, issue 11, 119-121.