ON AN EXPANSION FORMULA FOR THE MULTIVARIABLE I - FUNCTION INVOLVING GENERALIZED LEGENDRE’S ASSOCIATED FUNCTION

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ABSTRACT

The authors have established a new expansion formula for multivariable I-function due to Prasad [5] in terms of products of the multivariable I-function and the generalized Legendre’s associated function due to Meulenbeld [3]. Some special cases are given in the last.

Keywords: Multivariable I-function, Generalized Legendre’s associated function, Multivariable H-function.

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1. INTRODUCTION

The multivariable I-function introduced by Prasad [5] will be define and represent it in the following manner:

\[
I\left[ z_1, \ldots, z_r \right] = \int_{\Omega} I^{0,a_1,\ldots,a_{2m};a',\ldots,a'}{\varphi}(a_1, a_2, \ldots, a_{2m}; a', \ldots, a') \prod_{j=1}^{r} \left( b_j^q, d_j^r \right)
\]

\[
= \frac{1}{(2\pi i)^r} \int_{L} \prod_{r} \phi(s_r) \psi(s_1, \ldots, s_r) z_1^{s_1} \cdots z_r^{s_r} ds_1 \cdots ds_r
\]

(1.1)
Where

\[
 w = \sqrt{-1} 
\]

\[
 \phi(s_i) = \frac{\prod_{j=1}^{p_i} \Gamma(b_j^{(i)} - \beta_j^{(i)} s_i) \prod_{j=1}^{q_i} \Gamma(1 - a_j^{(i)} + \alpha_j^{(i)} s_i)}{\prod_{j=n_i+1}^{r} \Gamma(1 - b_j^{(i)} + \beta_j^{(i)} s_i) \prod_{j=1}^{r} \Gamma(a_j^{(i)} - \alpha_j^{(i)} s_i)} \quad \forall i \in (1, 2, ..., r) \]  

(1.2)

\[
 \psi(s_1, ..., s_r) = \frac{\prod_{j=1}^{p_1} \Gamma(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} s_i) \prod_{j=1}^{q_1} \Gamma(1 - a_j + \sum_{i=1}^{3} \alpha_j^{(i)} s_i)}{\prod_{j=n_2+1}^{r} \Gamma(a_j - \sum_{i=1}^{r} \beta_j^{(i)} s_i) \prod_{j=1}^{r} \Gamma(a_j - \sum_{i=1}^{3} \alpha_j^{(i)} s_i)} 
\]

\[
 \frac{\prod_{j=1}^{n} \Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} s_i)}{\prod_{j=n+1}^{r} \Gamma(a_j - \sum_{i=1}^{q} \beta_j^{(i)} s_i) \prod_{j=1}^{q} \Gamma(1 - b_j - \sum_{i=1}^{r} \beta_j^{(i)} s_i)} \]  

(1.3)

\[
 \alpha_j^{(i)}, \beta_j^{(i)}, \alpha_j^{(i)}(i = 1, ..., r)(k = 1, ..., r) \] are positive numbers,

\[
 a_j^{(i)}, b_j^{(i)}(i = 1, ..., r), a_{k_j}, b_{k_j}(k = 2, ..., r) \] are complex numbers and here

\[
 m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)}(i = 1, ..., r), n_k, p_k, q_k(k = 2, ..., r) \] are non-negative integers where

\[
 0 \leq m^{(i)} \leq q^{(i)}, 0 \leq n^{(i)} \leq p^{(i)}, q^k \geq 0, 0 \leq n_k \leq p_k. \] Here (i) denotes the numbers of dashes. The

contours \( L_i \) in the complex \( s_i \)-plane is of the Mellin-Barnes type which runs from \(-\infty\) to \(+\infty\) with indentations, if necessary, to ensure that all the poles of \( \Gamma(b_j^{(i)} - \beta_j^{(i)} s_i)(j = 1, ..., m^{(i)}) \) are

separated from those of \( \Gamma(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} s_i)(j = 1, ..., n) \).

For further details and asymptotic expansion of the \( I \)-function one can refer by Prasad [5].

In what follows, the multivariable \( I \)-function defined by Prasad [5] will be represented in the contracted notation:

\[
 I_{0, n_2; ...; 0, n_r, \{m^{(i)}, n^{(i)}\} ... \{m^{(r)}, n^{(r)}\}}^{p_1, q_2; ...; p_r, q_r, \{p^{(i)}, q^{(i)}\} ... \{p^{(r)}, q^{(r)}\}}[z_1, ..., z_r] 
\]

Or simply by \( I[z_1, ..., z_r] \).

According to the asymptotic expansion of the gamma function, the counter integral (1.1) is absolutely convergent provided that
\[ |\arg z_i| < \frac{1}{2} \pi U_i, U_j > 0 \quad ; \quad i = 1, 2, \ldots, r \quad (1.4) \]

Where

\[ U_i = \sum_{j=1}^{d} \alpha_{j}^{(i)} - \sum_{j=n^{(i)}+1}^{d} \alpha_{j}^{(i)} + \sum_{j=1}^{n^{(i)}} \beta_{j}^{(i)} - \sum_{j=m^{(i)}+1}^{n^{(i)}} \beta_{j}^{(i)} \]

\[ + (\sum_{j=1}^{n} \alpha_{2j}^{(i)} - \sum_{j=n+1}^{n} \alpha_{2j}^{(i)} ) + (\sum_{j=1}^{n} \alpha_{3j}^{(i)} - \sum_{j=n+1}^{n} \alpha_{3j}^{(i)} ) \]

\[ + \ldots + (\sum_{j=1}^{n} \alpha_{ij}^{(i)} - \sum_{j=n+1}^{n} \alpha_{ij}^{(i)} ) \]

\[ - (\sum_{j=1}^{n} \beta_{2j}^{(i)} + \sum_{j=1}^{n} \beta_{3j}^{(i)} + \ldots + \sum_{j=1}^{n} \beta_{n(j)}^{(i)} ) \quad (1.5) \]

The asymptotic expansion of the \( I \)-function has been discussed by Prasad [5]. His results run as follow:

\[ I[z_1, \ldots, z_r] = 0(|z_1|^{a_1} \ldots |z_r|^{a_r}), \max\{|z_1|, \ldots, |z_r|\} \to 0 \]

Where

\[ \alpha_i = \min \Re(\beta_j^{(i)} / \beta_j^{(i)}), \quad j = 1, \ldots, m^{(i)} ; \quad i = 1, \ldots, r \quad (1.6) \]

And \( I[z_1, \ldots, z_r] = 0(|z_1|^{b_1} \ldots |z_r|^{b_r}), \min\{|z_1|, \ldots, |z_r|\} \to \infty \)

Where

\[ \beta_i = \max \Re\left(\frac{\alpha_j^{(i)} - 1}{\alpha_j^{(i)}}\right) ; \quad j = 1, \ldots, n^{(i)} , \quad i = 1, \ldots, r \quad (1.7) \]

The details of the function can be found in the paper of Prasad [5].

In this paper we will evaluate an integral involving generalized associated Legendre’s function and the multivariable \( I \)-function due to Prasad [5] and apply it in deriving an expansion for the multivariable \( I \)-function in series of products of associated Legendre’s function and the multivariable \( I \)-function.

2. THE INTEGRAL

The integral to be evaluated is:
On evaluating the type integral (1.1) and inverting the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, the value of the integral

\[
I = \int_{0}^{1} \left( 1-x \right)^{\rho-u} \left( 1+x \right)^{\sigma+v} P_{k-u}^{m,n}(x) \, dx
\]

is valid under the following set of conditions:

(i) \((\alpha_i, \beta_i) > 0; \forall i \in \{1, 2, \ldots, r\}; k - \frac{\mu-v}{2} \) is a positive integer, \(k \) is an integer \(\geq 0\).

(ii) \(\text{Re} \left( \rho - u + \sum_{i=1}^{r} \alpha_i b_j^{(i)} / \beta_j^{(i)} \right) < -1; \text{Re} \left( \sigma + v + \sum_{i=1}^{r} \beta_i b_j^{(i)} / \beta_j^{(i)} \right) > -1; (j = 1, 2, \ldots, m; i = 1, 2, \ldots, r)\)

And the conditions given in (1.4) to (1.7) are also satisfied.

**Proof:** On expressing the multivariable \(I\) -function in the integrand as a multiple Mellin-Barnes type integral (1.1) and inverting the order of integrations, which is justified due to the absolute convergence of the integrals involved in the process, the value of the integral

\[
= \left( 2\pi w \right)^r \prod_{I_{r}} \int_{L_{r}} \left( s_{r}, \ldots, s_{r} \right) \sum_{s_{i} \in \mathbb{C}} \left\{ \phi(s_{i}) z_{i}^{\xi} \right\}
\]

is valid under the following set of conditions:

(i) \((\alpha_i, \beta_i) > 0; \forall i \in \{1, 2, \ldots, r\}; k - \frac{\mu-v}{2} \) is a positive integer, \(k \) is an integer \(\geq 0\).

(ii) \(\text{Re} \left( \rho - u + \sum_{i=1}^{r} \alpha_i b_j^{(i)} / \beta_j^{(i)} \right) < -1; \text{Re} \left( \sigma + v + \sum_{i=1}^{r} \beta_i b_j^{(i)} / \beta_j^{(i)} \right) > -1; (j = 1, 2, \ldots, m; i = 1, 2, \ldots, r)\)

On evaluating the \(x\) -integral with the help of the integral ([4], p.343, eq. (38)):
\[ 2^\rho^\sigma \frac{m-n}{2} \Gamma\left(\frac{\rho - m}{2} + 1\right) \Gamma\left(\sigma + \frac{n}{2} + 1\right) \]

\[ \Gamma(1-m) \Gamma\left(\rho + \sigma - \frac{m-n}{2} + 2\right) \]

\[ \times_3 F_2 \left[ -k, n-m+k+1, \rho - \frac{m}{2} + 1; 1-m, \rho - \sigma - \frac{m-n}{2} + 2; 1 \right] \quad (2.2) \]

Provided that \( \text{Re} \left( \frac{\rho - m}{2} \right) > -1; \text{Re} \left( \frac{\sigma + n}{2} \right) > -1 \) and interpreting the result with the help of (1.1), the integral (2.1) is established.

### 3. EXPANSION THEOREM

Let the following conditions be established:

(i) \( \beta_1, \ldots, \beta_r > 0; \alpha_1, \ldots, \alpha_r > 0 \) (or \( \beta_1, \ldots, \beta_r \geq 0; \alpha_1, \ldots, \alpha_r > 0 \));

(ii) \( m^{(i)}, n^{(i)}, p^{(i)}, q^{(i)} (i = 1, \ldots, r), n_k, p_k, q_k (k = 2, \ldots, r) \) are non-negative integers where \( 0 \leq m^{(i)} \leq q^{(i)}, 0 \leq n^{(i)} \leq p^{(i)}, q_k \geq 0, 0 \leq n_k \leq p_k \) and the conditions given by (1.4) to (1.7) are also satisfied.

(iii) \( \text{Re}(u) > -1, \text{Re}(v) > -1, \text{Re} \left( \rho - u + \sum_{i=1}^{r} \alpha_i \frac{b_i^{(i)}}{\beta_j^{(i)}} \right) > -1; \)

\( \text{Re} \left( \sigma + v + \sum_{i=1}^{r} \beta_i \frac{b_i^{(i)}}{\beta_j^{(i)}} \right) > -1; (j = 1, 2, \ldots, m_i; i = 1, 2, \ldots, r). \)

Then the following expansion formula holds:

\[ (1-x)^{\rho-u} (1-x)^{\sigma-v} I \left[ (1-x)^{\alpha_1} (1+x)^{\beta_1} z_1, \ldots, (1-x)^{\alpha_r} (1+x)^{\beta_r} z_r \right] \]

\[ = 2^{\rho+\sigma} \sum_{N=0}^{\infty} \frac{(2N-u+v+1)\Gamma(N-u+1)\Gamma(1+v-u+N+\mu)(-N)_{\mu}}{N! \mu! \Gamma(1+v+N)\Gamma(1-u+\mu)} \]

\[ P_{N-u-v}^{\mu,v}(x) I_{N-u-v}^{N} \]

\[ = 2^{\rho+\sigma} \sum_{N=0}^{\infty} \frac{N!}{N!} \prod_{i=0}^{r} \Gamma(m_i^{(i)}) \prod_{j=0}^{r} \Gamma(n_j^{(j)}) \prod_{k=0}^{r} \Gamma(p_k^{(k)}) \prod_{l=0}^{r} \Gamma(q_l^{(l)}) \]

\[ \left[ \begin{array}{c}
2^{n_1 \alpha_1} z_1 \\
\vdots \\
2^{n_r \alpha_r} z_r \\
\end{array} \right] \left( a_1, \beta_1 ; \alpha_2, \beta_2 \right) \frac{1}{h_{a_1} \cdots (1-v; \beta_1)_{a_1}} \]

\[ + \left[ \begin{array}{c}
2^{n_1 \beta_1} z_1 \\
\vdots \\
2^{n_r \beta_r} z_r \\
\end{array} \right] \left( b_1, \beta_1 ; \beta_2 \right) \frac{1}{h_{b_1} \cdots (1-v; \beta_1)_{b_1}} \]
\[
\begin{align*}
(u-v-\mu \pi_1, ..., \pi_r, \alpha_1, \alpha_1, ..., \alpha_1, \beta_1, ..., \beta_1, \gamma_1, ..., \gamma_1, \lambda_1, ..., \lambda_1, \mu_1, ..., \mu_1) \\
(u-v-\sigma-\mu-1, \pi_1, ..., \pi_r, \alpha_1, \alpha_1, ..., \alpha_1, \beta_1, ..., \beta_1, \gamma_1, ..., \gamma_1, \lambda_1, ..., \lambda_1, \mu_1, ..., \mu_1) \\
(u-v-\sigma-\mu-1, \pi_1, ..., \pi_r, \alpha_1, \alpha_1, ..., \alpha_1, \beta_1, ..., \beta_1, \gamma_1, ..., \gamma_1, \lambda_1, ..., \lambda_1, \mu_1, ..., \mu_1)
\end{align*}
\]

(3.1)

**Proof:** Let

\[
f(x) = (1-x)^{\mu/2}(1+x)^{\nu/2}I[(1-x)^{\alpha_1}(1+x)^{\beta_1}z_1, ..., (1-x)^{\alpha_r}(1+x)^{\beta_r}z_r]
\]

\[
= \sum_{N=0}^{\infty} C_N P_{N,u,v}^{\mu,\nu}(x)
\]

(3.2)

Equation (3.2) is valid since \( f(x) \) is continuous and of bounded variation in the interval \((-1,1)\).

Now, multiplying both the sides of (3.2) by \( P_{N,u,v}^{\mu,\nu}(x) \) and integrating with respect to \( x \) from -1 to 1; evaluating the L.H.S. with the help of (2.1) and on the R.H.S. interchanging the order of summation, using ([2],p.176, eq. (75)) and then applying orthogonality property of the generalized Legendre’s associated functions ([4],p.340, eq.(27)):

\[
\int_{-1}^{1} P_{N,u,v}^{\mu,\nu}(x) P_{N',u,v}^{\mu,\nu}(x) dx
\]

\[
= \begin{cases} 
0; & \text{if } k \neq N \\
\frac{2^{\mu-1}k\Gamma(k+1)}{(2k+1)(k+1)\Gamma(k-u+1)\Gamma(k-v+1)} & \text{if } k = N
\end{cases}
\]

(3.3)

Provided that \( \text{Re}(\mu), \text{I}(\mu) < 1 \); we obtain

\[
C_k = \frac{2^{\mu-\sigma}(2k-u+v+1)\Gamma(k-u+1)}{k!\Gamma(k+v+1)} \sum_{\mu=0}^{k} (-k)_\mu \Gamma(k-u+v+\mu+1) \\
\left[ \sum_{m=0}^{\infty} (n^\mu,m^\mu) \left[ \left( \begin{array}{c} \alpha_1, \alpha_1, ..., \alpha_1, \beta_1, ..., \beta_1, \gamma_1, ..., \gamma_1, \lambda_1, ..., \lambda_1, \mu_1, ..., \mu_1 \\ n, m \end{array} \right) \right] \right]
\]

(3.4)

Now on substituting the values of \( C_k \) in (3.2), the result follows.

### 4. SPECIAL CASES

If in (2.1), we put \( n_2 = ... = n_{r-1} = 0 = p_2 = ... = p_{r-1}, q_2 = ... = q_{r-1} = 0 \), the multivariable \( I \)-function reduces to multivariable \( H \)-function and we get result given by Saxena and Ramawat [6]
\[(1-x)^{\rho/u} (1+x)^{\sigma/v} H \left[ (1-x)^{\alpha_1} (1+x)^{\beta_1} z_1, \ldots, (1-x)^{\alpha_N} (1+x)^{\beta_N} z_N \right] \]

\[= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N-u+v+1) \Gamma(N-u+1) \Gamma(1+v-u+N+\mu)(-N)_{\mu}}{N! \mu! \Gamma(1+v+N) \Gamma(1-u+\mu)} \cdot P^{\mu,v}_{N-u-v/2}(x) H^{0,s+2(m,n);\ldots\left(m^{(r)},n^{(r)}\right)}_{\rho+2,q+1(p,q);\ldots\left(p^{(r)},q^{(r)}\right)} \]

\[= \left[ (u-v-r,\ldots,\alpha_{ij},\alpha_{ij},\ldots,\beta_{ij},\beta_{ij},\ldots)_{i,j} \right] \]

\[(4.1)\]

Provided all the conditions given with (3.1) and the conditions ([7], p.252-253, eq. (c.4), (c.5) and (c.6)) are satisfied.

For \(n = 0 = p, q = 0\), the multivariable \(H\) -function breaks up into a product of \(r\) \(H\) -function and consequently, (4.1) reduces to

\[(1-x)^{\rho/u} (1+x)^{\sigma/v} \prod_{i=1}^{r} \left[ \left(1-x\right)^{\alpha_i} (1+x)^{\beta_i} \left\{ \left(\alpha_i^{(r)},\alpha_i^{(r)}\right)_{i=1}^{n_i} \right\} \right] \]

\[= 2^{\rho+\sigma} \sum_{N=0}^{\infty} \sum_{\mu=0}^{N} \frac{(2N-u+v+1) \Gamma(N-u+1) \Gamma(1+v-u+N+\mu)(-N)_{\mu}}{N! \mu! \Gamma(1+v+N) \Gamma(1-u+\mu)} \cdot P^{\mu,v}_{N-u-v/2}(x) H^{0,s+2(m,n);\ldots\left(m^{(r)},n^{(r)}\right)}_{\rho+2,q+1(p,q);\ldots\left(p^{(r)},q^{(r)}\right)} \]

\[(4.2)\]

For \(r = 1\), (4.2) gives riseto the result due to Anandani [1].

5. REFERENCES


